CONSTRAINED PROBLEMS

- Standard Mathematical Statement
  - ***Minimize*** $f(\mathbf{x}) = f(x_1, x_2, \ldots\ldots, x_n)$
  - ***subject to***

$$h_k(\mathbf{x}) = 0 \qquad k = 1, \ldots\ldots, n_e$$
$$g_j(\mathbf{x}) \le 0 \qquad j = 1, \ldots\ldots, n_g$$

- Necessary Conditions: Equality Constrained Problems
  - One equality with two design variables

Minimize $f(\mathbf{x}) = f(x_1, x_2)$ ***subject to*** $h(x_1, x_2) = 0$

- Necessary Conditions for Functions with Equality Constraints:
  - Form a Lagrange function defined as

$$L(\boldsymbol{x}, \boldsymbol{v}) = f(\boldsymbol{x}) + \sum_{k=1}^{n_e} v_k h_k(\boldsymbol{x}) = f(\boldsymbol{x}) + \boldsymbol{v}^T \boldsymbol{h}(\boldsymbol{x})$$

  - Necessary conditions for candidate minimum point ***x****

$$\nabla L(\boldsymbol{x}^*, \boldsymbol{v}^*) = \boldsymbol{0} \qquad \frac{\partial L(\boldsymbol{x}^*, \boldsymbol{v}^*)}{\partial x_i} = 0 \qquad i = 1, \ldots, n$$

$$\frac{\partial L(\boldsymbol{x}^*, \boldsymbol{v}^*)}{\partial v_k} \equiv h_k(\boldsymbol{x}^*) = 0 \qquad k = 1, \ldots, n_e$$

  - The first of the necessary conditions can be rearranged

$$\frac{\partial f(\boldsymbol{x}^*)}{\partial x_i} = -\sum_{k=1}^{n_e} v^*_k \frac{\partial h_k(\boldsymbol{x}^*)}{\partial x_i} \qquad i = 1, \ldots, n$$

$$\nabla f(\boldsymbol{x}^*) = v_1 \nabla h_1(\boldsymbol{x}^*) + v_2 \nabla h_2(\boldsymbol{x}^*) + \ldots\ldots + v_{n_e} \nabla h_{n_e}(\boldsymbol{x}^*)$$

- Necessary Conditions for General Constrained Problems:
  - For inequality constraints introduce slack variables ***s***

$$g_j(\mathbf{x}) \le 0 \qquad g_j(\mathbf{x}) + s^2_j = 0 \qquad j = 1, \ldots\ldots, n_g$$

  - Lagrange function for eq. and ineq. constraints

$$L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \sum_{k=1}^{n_e} v_k h_k(\mathbf{x}) + \sum_{j=1}^{n_g} u_j(g_j(\mathbf{x}) + s^2_j) = f(\mathbf{x}) + \mathbf{v}^T\mathbf{h}(\mathbf{x}) + \mathbf{u}^T(\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)$$

  - Kuhn-Tucker necessary conditions for eq. and ineq. constraint

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{k=1}^{n_e} v^*_k \frac{\partial h_k}{\partial x_i} + \sum_{j=1}^{n_g} u^*_j \frac{\partial g_j}{\partial x_i} = 0$$

$$i = 1, \ldots, n$$

$$j = 1, \ldots, n_g$$

$$k = 1, \ldots, n_e$$

$$h_k(\boldsymbol{x}^*) = 0$$

$$g_j(\boldsymbol{x}) + s^2_j = 0 \qquad u_j^* s_j = 0 \qquad u_j^* \geq 0$$

- Post Optimality Analysis:
  - Assume that minimization problem is solved with $h_i(\mathbf{x}^*)=0$ and $g_j(\mathbf{x}^*) \le 0$.
  - Consider the modified problem where b_i and e_j are small variations

$$h_i(\mathbf{x}) = b_i \qquad i = 1, \ldots, n_e$$
$$g_j(\mathbf{x}) \le e_j \qquad j = 1, \ldots, n_g$$

  - The optimum for the perturbed problem $\mathbf{x}^*= \mathbf{x}^*(\mathbf{b},\mathbf{e})$ and we want $f^*(\mathbf{b},\mathbf{e})$
  - Derivatives of the cost function w.r.t. the right hand side parameters

$$\frac{\partial f(\mathbf{x}^*(\mathbf{0},\mathbf{0}))}{\partial b_i} = -v_i^* \qquad i = 1, \ldots, n_e \qquad\qquad \frac{\partial f(\mathbf{x}^*(\mathbf{0},\mathbf{0}))}{\partial e_j} = -u_j^* \qquad j = 1, \ldots, n_g$$

- The perturbed value of the cost function

$$\Delta f = -\sum_{i} v_i^* b_i - \sum_{j} u_j^* e_j$$